Exercise 1 *Let the initial value problem,*

$$\begin{cases} y'(t) = f(t, y(t)), & t \in [a, b], \\ y(a) = y_0 \in \mathbb{R}, \end{cases}$$

where $f : [a, b] \times \mathbb{R} \to \mathbb{R}$ has the sufficiently regularity for its derivatives, i.e., every derivative of f that we may use, is bounded. Let $\{Y^n\}$, $n \geq 0$, the grid function that approximates $\{y(t^n)\}$, $n = 0, 1, \ldots, M$, $M \in \mathbb{N}$ and it produced from Runge-Kutta method of $r-$stages, with fixed mesh step $h = \frac{b-a}{M}$, i.e.,

$$\begin{aligned} Y^0 &:= y_0 \\ Y^{n,s} &= Y^{n-1} + h \sum_{j=1}^{r} \lambda_{sj} f(t^{n,j}, Y^{n,j}), \quad 1 \leq s \leq r, \\ Y^n &= Y^{n-1} + h \sum_{s=1}^{r} \alpha_s f(t^{n,s}, Y^{n,s}), \qquad n = 1, 2, \ldots, M, \end{aligned}$$

Prove that there exists a constant $C > 0$, independent of h, such that

1.
$$\max_{n,s} |y(t^{n,s}) - \xi^{n,s}| \leq Ch.$$

2.
$$\max_{n} |y(t^{n,s}) - \xi^{n,s}| \leq Ch^2 \Leftrightarrow \mu_s = \sum_{j=1}^{r} \lambda_{sj}, \quad \forall\, s = 1, \ldots, r,$$

where the grid functions for $s = 1, \ldots, r$, $\{\xi^{n,s}\}$, $n = 1, \ldots, M$, are defined as

$$\xi^{n,s} = y(t^{n-1}) + h \sum_{j=1}^{r} \lambda_{sj} f(t^{n,j}, y(t^{n,j})), \quad 1 \leq s \leq r,$$

Solution We may prove both 1., 2., simultaneously, i.e., let for all $s = 1, \ldots, r$, subtract from $\xi^{n,s}$, the $y(t^{n,s})$, to get

$$\xi^{n,s} - y(t^{n,s}) = y(t^{n-1}) - y(t^{n,s}) + h \sum_{j=1}^{r} \lambda_{sj} f(t^{n,j}, y(t^{n,j})), \quad 1 \leq s \leq r.$$

Then, from Taylor expansion and the definition of $t^{n,s} := t^{n-1} + \mu_s\, h$, $s = 1, \ldots, r$, we get

$$\begin{aligned} y(t^{n-1}) - y(t^{n,s}) &= y(t^{n-1}) - y(t^{n-1} + \mu_s\, h) \\ &= y(t^{n-1}) - \left(y(t^{n-1}) + h\mu_s y'(t^{n-1}) + \frac{h^2 \mu_s^2}{2!} y''(t^{n-1}) + \frac{h^3 \mu_s^3}{3!} y'''(\tau) \right) \\ &= -h\mu_s y'(t^{n-1}) - \frac{h^2 \mu_s^2}{2!} y''(t^{n-1}) - \frac{h^3 \mu_s^3}{3!} y'''(\tau), \quad \tau \in (t^{n-1}, t^{n,s}). \end{aligned}$$

So,

$$\xi^{n,s} - y(t^{n,s}) = -h\mu_s y'(t^{n-1}) - \frac{h^2 \mu_s^2}{2!} y''(t^{n-1}) - \frac{h^3 \mu_s^3}{3!} y'''(\tau) + h \sum_{j=1}^{r} \lambda_{sj} f(t^{n,j}, y(t^{n,j})), \quad 1 \leq s \leq r.$$

Noticing that for all $j = 1, \ldots, r$,

$$f(t^{n,j}, y(t^{n,j})) = y'(t^{n,j}) = y'(t^{n-1} + \mu_j\, h) = y'(t^{n-1}) + \mu_j\, hy''(\zeta), \quad \zeta \in (t^{n-1}, t^{n,j})$$

Therefore, for all $1 \leq s \leq r$,

$$\begin{aligned} \xi^{n,s} - y(t^{n,s}) &= -h\mu_s y'(t^{n-1}) - \frac{h^2 \mu_s^2}{2!} y''(t^{n-1}) - \frac{h^3 \mu_s^3}{3!} y'''(\tau) + h \sum_{j=1}^{r} (\lambda_{sj} y'(t^{n-1}) + \mu_j\, hy''(\zeta)), \\ &= hy'(t^{n-1}) \left(\sum_{j=1}^{r} \lambda_{sj} - \mu_s \right) + \mathcal{O}(h^2). \end{aligned}$$

So, $\xi^{n,s} - y(t^{n,s}) = \mathcal{O}(h^2)$ if and only if $\mu_s = \sum_{j=1}^{r} \lambda_{sj}$. Since these identities holds for all s, n, we may conclude for the maximum.

$\square$

Exercise 2 *Let the initial value problem,*

$$\begin{cases} y'(t) = f(t, y(t)), & t \geq 0, \\ y(0) = y_0 \in \mathbb{R}, \end{cases}$$

where $f : [a, b] \times \mathbb{R} \to \mathbb{R}$ has the sufficiently regularity for its derivatives, i.e., every derivative of f that we may use, is bounded.

Prove that trapezoidal rule has order 2.

Solution Let $\{Y^n\}$, $n \geq 0$, the grid function that approximates $\{y(t^n)\}$, $n \geq 0$, and it produced from trapezoidal rule with fixed mesh step $h > 0$, i.e.,

$$Y^0 := y_0, \quad Y^n = Y^{n-1} + h\frac{1}{2}f(t^{n-1}, Y^{n-1}) + h\frac{1}{2}f(t^n, Y^n), \quad n \geq 1.$$

The truncation error T, that passes through the point (t, y) where y is the exact solution of IVP, is defined as

$$T(t, y; h) := \Phi(t, y; h) - \frac{1}{h}(u(t + h) - u(t)),$$

where $u(t)$ is the reference solution that passes through the point (t, y), and defined as

$$\begin{cases} u'(x) = f(x, u(x)), & x \in [t, t + h], \\ u(t) = y(t). \end{cases}$$

The increment function for a ($r-$stage) RKM with Butcher tableu

$$\begin{array}{c|c} \mu & \Lambda \\ \hline & \alpha^T \end{array},$$

where $\Lambda \in \mathbb{R}^{r,r}$, $\mu, \alpha \in \mathbb{R}^r$, is defined as

$$\Phi(t, y; h) := \sum_{s=1}^{r} \alpha_s k_s(t, y; h), \quad \text{with} \quad k_s(t, y; h) := f(t + \mu_s h, y + h\sum_{j=1}^{r} \lambda_{sj} k_s(t, y; h)).$$

Notice that the Butcher tableu of trapezoidal rule is

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & \frac{1}{2} & \frac{1}{2} \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array},$$

so $r = 2$ and

$$\Phi(t, y; h) := \frac{1}{2}k_1(t, y; h) + \frac{1}{2}k_2(t, y; h),$$

with

$$\begin{aligned} k_1(t, y; h) &= f(t, y) \\ k_2(t, y; h) &= f(t + h, y + \frac{1}{2}hk_1 + \frac{1}{2}hk_2). \end{aligned}$$

A method is of $p \geq 1$ order if

$$T(t, y; h) := \tau(t, y)h^p + \mathcal{O}(h^{p+1}), \quad \text{with} \quad \tau(t, y) \neq 0.$$

To prove the second order accuracy for trapezoidal rule, first to simplify the notation let $f := f(t, y)$, $f_t := f_t(t, y)$, $f_y := f_y(t, y)$ and

$$\begin{aligned} I_1 &= \Phi(t, y; h) \\ I_2 &= \frac{1}{h}(u(t + h) - u(t)). \end{aligned}$$

For I_1, we need to expand it as $I_1 = \tau_1 + \mathcal{O}(h^p)$. So,

$$I_1 = \Phi(t, y; h) = \frac{1}{2}k_1(t, y; h) + \frac{1}{2}k_2(t, y; h),$$

with

$$k_1(t, y; h) = f$$

and

$$\begin{aligned} k_2(t, y; h) = f(t + h, y + \frac{1}{2}hk_1 + \frac{1}{2}hk_2) &= f + hf_t + \frac{1}{2}(hk_1 + hk_2)f_y + \mathcal{O}(h^2) \\ &= f + hf_t + \frac{1}{2}(hf + h(f + \mathcal{O}(h)))f_y + \mathcal{O}(h^2), \end{aligned}$$

were we have used the fact that $k_1 = f$ and $k_2 = f + \mathcal{O}(h)$, i.e., we expand again the k_2 with Taylor. All in all,

$$k_2(t, y; h) = f + hf_t + hf_y f + \mathcal{O}(h^2),$$

and therefore,

$$I_1 = \frac{1}{2}f + \frac{1}{2}f + h\frac{1}{2}(f_t + f_y f) + \mathcal{O}(h^2)$$

For I_2, we need to expand it as $I_2 = \tau_2 + \mathcal{O}(h^{p+1})$. So,

$$\begin{aligned} I_2 &= \frac{1}{h}(u(t + h) - u(t)) = \frac{1}{h}(u(t) + hu'(t) + \frac{h^2}{2!}u''(t) + \frac{h^3}{3!}u'''(\xi) - u(t)) \\ &= u'(t) + \frac{h}{2!}u''(t) + \frac{h^2}{3!}u'''(\xi), \end{aligned}$$

where $\xi \in (t, t + h)$. Notice, that

$$\begin{aligned} u'(t) &= f(t, u(t)) = f(t, y(t)) = f \\ u''(t) &= (u'(t))' = f_t + f_y f. \end{aligned}$$

Thus,

$$I_2 = f + \frac{h}{2}(f_t + f_y f) + \mathcal{O}(h^2).$$

Therefore,

$$T(t, y; h) = I_1 - I_2 = \mathcal{O}(h^2).$$

□